§6 Chain Condition

太一般了，特殊一些（加一些有限性条件）

$(\Sigma, \leq)$ = partially ordered set.

- reflexive $x \leq x$
- transitive $x \leq y, y \leq z \Rightarrow x \leq z$
- $x \leq y$ & $y \leq x \Rightarrow x = y$.

Prop 6.1: $(\Sigma, \leq)$ = p.o.s.

i) $\forall\ x_1 \leq x_2 \leq \cdots$ is stationary (i.e. $\exists\ n \gg 0$, s.t. $x_n = x_{n+1} = \cdots$)

ii) $\forall$ non-empty subset of Σ has maximal element

Pf: i) $\Rightarrow$ ii). If ii) is false. $\exists\ T \subseteq \Sigma$ has no maximal elements.

inductively $\Rightarrow \exists\ x_1 \leq x_2 \leq \cdots$ with $x_i \neq x_j$. ↯

ii) $\Rightarrow$ i): $\{x_m\}_{m \geq 1}$ has maximal element x_n for some n

Let M be a module.

$$\Sigma_M := \{ N \subseteq M \mid \text{submodule} \}$$

- A module M satisfies <u>ascending chain condition (a.c.c)</u> if the condition i) in Prop 6.1 holds for $(\Sigma_M, \subseteq)$.

- A module M satisfies <u>maximal condition</u> if the condition ii) in Prop 6.1 holds for $(\Sigma_M, \subseteq)$.

  in this case, M is called Noetherian

- A module M satisfies <u>descending chain condition (a.c.c)</u> if the condition i) in Prop 6.1 holds for $(\Sigma_M, \supseteq)$.

- A module M satisfies <u>minimal condition</u> if the condition ii) in Prop 6.1 holds for $(\Sigma_M, \supseteq)$.

  in this case, M is called Artinian.

Example : i) (a.c.c. d.c.c) f. abel. gp

ii) (a.c.c, ~~d.c.c~~) $\mathbb{Z}$ (group),

iii) (~~a.c.c~~ d.c.c) $\mathbb{Q}_p/\mathbb{Z}_p \cong \{ \frac{n}{p^m} \mid n \in \mathbb{Z}, m \in \mathbb{N} \} / \mathbb{Z} \subseteq \mathbb{Q}/\mathbb{Z}$

iv) (~~a.c.c~~, ~~d.c.c~~) $\mathbb{Z}_{(p)} = \{ \frac{n}{p^m} \mid n \in \mathbb{Z}, m \in \mathbb{N} \} \subseteq \mathbb{Q}$

a.c.c & d.c.c on ideals

v) (a.c.c, ~~d.c.c~~) $\quad k[x]$

vi) (~~a.c.c~~, ~~d.c.c~~) $\quad k[x_1, x_2, \dots]$

vii) (a.c.c, d.c.c) $\quad$ field

viii) (~~a.c.c~~, d.c.c) $\quad$?

d.c.c on ideals $\Rightarrow$ a.c.c on ideals (later)

Prop 6.2 $\quad M = $ Noetherian (A-mod) $\Leftrightarrow \forall N \subseteq M$ f.g.

submodule

Pf : $\Rightarrow$) Suppose not. We may assume N is not f.g.

Inductively $\forall x_1, \dots, x_n \quad \exists x_{n+1} \in N \setminus \sum_{i=1}^{n} A \cdot x_i$

$$N_n := \sum_{i=1}^{n} A x_i \subseteq N$$

$$\Rightarrow N_1 \subsetneq N_2 \subsetneq N_3 \subsetneq \cdots \subsetneq N_n \subsetneq \cdots \quad ↯$$

$\Leftarrow$) $\forall M_1 \subseteq M_2 \subseteq \cdots \subseteq M$

$$N := \bigcup_{i=1}^{\infty} M_i \subseteq M \Rightarrow N = \sum_{j=1}^{n} A x_j$$

$$x_j \in M_{n_j} \Rightarrow N \subseteq N_n, \quad n = \max_j n_j.$$

$$\Rightarrow N_n = N_{n+1} = \cdots$$

Noetherian 更重要，Artin 简单.

Prop 6.3 $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ exact.

i) M = Noetherian $\Leftrightarrow$ M' & M'' = Noetherian

ii) M = Artinian $\Leftrightarrow$ M' & M'' = Artinian

Pf: i) $\Rightarrow$): $\forall\ M_1'' \subseteq M_2'' \subseteq \cdots \subseteq M''$

$\Rightarrow\ \beta^{-1}(M_1'') \subseteq \beta^{-1}(M_2'') \subseteq \cdots \subseteq \beta^{-1}(M'') = M$

$\Rightarrow\ \beta^{-1}(M_n'') = \beta^{-1}(M_{n+1}'') = \cdots$

$\Rightarrow\ M_n'' = M_{n+1}'' = \cdots$

$\forall\ M_1' \subseteq M_2' \subseteq \cdots \subseteq M' (\subseteq M) \Rightarrow \checkmark$

$\Leftarrow$): $\forall\ M_1 \subseteq M_2 \subseteq \cdots \subseteq M$

$\beta(M_1) \subseteq \beta(M_2) \subseteq \cdots \subseteq M''$

$\alpha^{-1}(M_1) \subseteq \alpha^{-1}(M_2) \subseteq \cdots \subseteq M'$

$\Rightarrow\ \beta(M_n) = \beta(M_{n+1}) = \cdots$ & $\alpha^{-1}(M_n) = \alpha^{-1}(M_{n+1}) = \cdots$

$0 \to \alpha^{-1}(M_i) \to M_i \to \beta(M_i) \to 0 \Rightarrow \checkmark$

Cor 6.4. M_i = Noetherian (resp. Artin) $\Rightarrow \bigoplus_{i=1}^{n} M_i$ = Noetherian (resp. Artin).

Pf: $0 \to M_n \to \bigoplus_{i=1}^{n} M_i \to \bigoplus_{i=1}^{n-1} M_i \to 0$ exact $\square$

A ring A is called to be Noetherian (resp. Artin), if it is so as an A-module. (i.e. a.c.c or d.c.c. on ideal)

Example: i) (Artin & Noetherian) field, $\mathbb{Z}/n\mathbb{Z}$

ii) (~~Artin~~ & Noetherian) $\mathbb{Z}$ (PID non-field)

iii) (~~Artin~~ & ~~Noetherian~~) $k[x_1, x_2, \ldots]$

- $C(X)$: X = compact infinite Hausdorff sp
  $C(X)$ = ring of cont. real functions

  $X \supset F_1 \supsetneq F_2 \supsetneq F_3 \supsetneq \cdots$ (closed subsets)

  $\mathfrak{a}_n := \{ f \in C(X) \mid f(F_n) = 0 \}$

  $\Rightarrow \mathfrak{a}_1 \subsetneq \mathfrak{a}_2 \subsetneq \cdots$

  $\Rightarrow C(X) \neq$ noetherian.

iv) (Artin & ~~Noetherian~~) ϕ

Prop 6.5 $\left.\begin{array}{l} A = \text{noetherian (resp. Artin)} \\ M = \text{f.g. over } A \end{array}\right\} \Rightarrow M = \text{noetherian (resp. Artin)}$

Pf: $0 \to \ker\pi \to \bigoplus_{i=1}^{n} A \xrightarrow{\pi} M \to 0$ $\square$

Prop 6.6 $\left.\begin{array}{l} A = \text{noetherian (resp. Artin)} \\ \mathfrak{a} \lhd A \end{array}\right\} \Rightarrow A/\mathfrak{a} = \text{noetherian (resp. Artin)}$

Pf (6.3) or (6.5) $\Rightarrow A/\mathfrak{a}$ = noetherian A-module

$\Rightarrow A/\mathfrak{a}$ = noetherian $A/\mathfrak{a}$-module

$\Rightarrow A/\mathfrak{a}$ = noetherian ring.

A chain of submodules of a module M is a sequence.

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_n = 0$$

length $:= n$

A composition series := maximal chain.

i.e. M_{i-1}/M_i = simple.

Prop 6.7 : Suppose M has a composition series of length n. Then

i) all comp. series has length n

ii) every chain can be extended to a comp. series

Pf: $\ell(M) := \begin{cases} \text{least length of a comp. series.} \\ \infty, \quad M \text{ has no comp. series.} \end{cases}$

i) $N \subsetneq M \Rightarrow \ell(N) < \ell(M)$

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_{\ell(M)} = 0$$

$$\Rightarrow N = N_0 \supset N \cap M_1 \supset N \cap M_2 \supset \cdots \supset N \cap M_{\ell(M)} = 0$$

$$N \cap M_i / N \cap M_{i+1} \hookrightarrow M_i / M_{i+1} \text{ : simple}$$

$$\Rightarrow N \cap M_i / N \cap M_{i+1} = \begin{cases} 0 \\ \text{simple} \end{cases} \Rightarrow \ell(M) \geq \ell(N)$$

Suppose $\ell(M) = \ell(N)$. Then

$$\frac{N \cap M_i}{N \cap M_{i+1}} \xrightarrow{\sim} M_i / M_{i+1}$$

$$N \cap M_i + M_{i+1} = M_i$$

$$\Rightarrow M = M_0 = N \cap M_0 + M_1$$
$$= N \cap M_0 + (N \cap M_1 + M_2)$$
$$= \cdots$$
$$= N \cap M_0 + N \cap M_1 + \cdots + N \cap M_{\ell(M)-1} + M_{\ell(M)}$$
$$= N \cap M_0 = N \quad \lightning$$

ii) Any chain in M has length $\leq \ell(M)$.

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_k = 0.$$

$$\Rightarrow \ell(M) > \ell(M_1) > \cdots > \ell(M_k)$$
$$\Rightarrow \ell(M) \geq k$$

iii) ii) $\Rightarrow$ all composition series have the same length.

Insert new terms until length is $\ell(M)$.

Prop 6.8. M has composition series $\Leftrightarrow$ it satisfies a.c.c & d.c.c

Pf: $\Rightarrow$) all chains are of bounded length $\Rightarrow$ ✓

$\Leftarrow$) $M_0 := M$

a.c.c. $\Rightarrow$ if $M_i \neq 0$. inductively

$\exists$ M_{i+1} = maximal among all proper submodules of M_i

$$M_0 \supsetneq M_1 \supsetneq M_2 \supsetneq \cdots$$

d.c.c. $\Rightarrow$ stop at some step i.e.

$M_n = 0$ for some n.

$$\Rightarrow \quad M_0 \supsetneq M_1 \supsetneq M_2 \supsetneq \cdots \supsetneq M_n = 0$$

is a composition series

<u>module of finite length</u> := module satisfying a.c.c. & d.c.c.

Fact (Jordan-Hölder theorem) $(M_i)_{0 \le i \le n}$ & $(M_i')_{0 \le i \le n}$

$$\Rightarrow (M_{i-1}/M_i)_{1 \le i \le n} \overset{\exists 1:1}{\longrightarrow} (M_{i-1}'/M_i')_{1 \le i \le n}$$

Prop 6.9: Length is additive on the class of all A-modules of finite length.

Pf: $\forall\ 0 \to M' \to M \to M'' \to 0$ exact

$M \supset M' \supset 0 \Rightarrow$

$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_k = M' \supsetneq M_{k+1} \supsetneq \cdots \supsetneq M_{k+l} = 0$

$l(M) = k + l, \quad l = l(M')$

$M_0/M' \supsetneq M_1/M' \supsetneq \cdots \supsetneq M_k/M' = 0$

& $\dfrac{M_i/M'}{M_{i+1}/M'} \cong M_i/M_{i+1}$ simple

$\Rightarrow l(M'') = k$

$\Rightarrow l(M) = k + l = l(M'') + l(M')$ $\square$.

Prop 6.10. k-vector space $=$ k-module (k = field). TFAE

i) f. dim

ii) f. length

iii) a.c.c

iv) d.c.c

v) Length = dimension.

Cor 6.11. A = ring. m_i = maximal. ideal of A $i = 1, \cdots, n$.

Suppose. $m_1 m_2 \cdots m_n = 0$, Then

A = noetherian $\Leftrightarrow$ A = Artin.

pf: $A \supseteq m_1 \supseteq m_1 m_2 \supseteq \cdots \supseteq m_1 m_2 \cdots m_n = 0$

$\Rightarrow$ $m_1 \cdots m_{i-1} / m_1 \cdots m_i$ = vector space of A/m_i

$$\text{a.c.c for } A \overset{(6.3)}{\Leftrightarrow} \text{a.c.c for } m_1 \cdots m_{i-1} / m_1 \cdots m_i \quad \forall i$$

$$\Updownarrow (6.11)$$

$$\text{d.c.c for } A \overset{(6.3)}{\Leftrightarrow} \text{d.c.c for } m_1 \cdots m_{i-1} / m_1 \cdots m_i \quad \forall i$$